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ON THE WELFARE SIGNIFICANCE OF NATIONAL
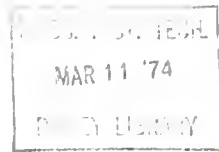PRODUCT IN A DYNAMIC ECONOMY

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On The Welfare Significance of National Product in a Dynamic Economy[1]

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1. Introduction

Repeatedly and unabashedly the national income statistician calculates a number he calls net national product by adding in the value of the nation's net investment to its consumption. What does this single figure measure? The usual welfare interpretation of index numbers can perhaps be used to excuse combining apples and oranges, but it falls short of providing an adequate justification for NNP. Economic activity has as its ultimate end consumption, not capital formation. The most complete inference which can be drawn from such a tenet is that investment must be treated as an intermediate good in a multi-period system whose final products are the various consumptions of different years. As Samuelson has convincingly argued,[2] the rigorous search for a meaningful welfare concept leads to a rejection of all current income concepts and ends up with something closer to a "wealth-like magnitude" such as the present discounted value of future consumption.

Actually, it is not really a question of choosing between a conventional but inappropriate current income concept and an impractical but correct wealth-like magnitude, because in principle they are merely different sides of the same coin. As I hope to show in this paper, the welfare

justification of net national product is just the idea that in theory it is a proxy for the present discounted value of future consumption.

2. A Formulation of the Problem

We abstract heroically in more ways than one.

First of all, for simplicity it is assumed that (in effect) there is just one composite consumption good. It might be calculated as an index number with given price weights, or as a multiple of some fixed basket of goods, or more generally as any cardinal utility function. The important thing is that the consumption level in period t can be un-ambiguously registered by the single number $C(t)$. Purging consumption of the index number problem will allow us to focus more sharply on the general meaning and significance of combining it with investment.

The notion of a capital good used in this paper is meant to be quite a bit more general than the usual equipment, structures, and inventories. Strictly speaking, pools of exhaustible natural resources ought to qualify as capital, and so should states of knowledge resulting from learning or research activities.[3] For convenience, all non-capital contributions to production are treated as fixed over time. In the case of labor, this assumption would always be satisfied so long as everything were calculated on a per-capita basis.[4] Note that in effect we are making the extreme

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[3] Included here would be so-called "human capital". Investment in learning or research is of course most easily calsulated on the cost side, since it obviates the necessity for evaluating increases in the semi-imaginary "stock of knowledge".

[4] A similar comment applies to non-attributable ("atmospheric") tech-nical change which is purely labor augmenting; in such a case every-thing should be calculated per unit of "effective" or "augmented" labor.

abstraction that <u>all</u> sources of economic growth have been identified and attributed to one or another form of capital, broadly defined.

Suppose that altogether there are n capital goods. The stock of capital of type i ($1 \leq i \leq n$) in existence at time t is denoted $K_i(t)$ and its net investment flow is $I_i(t) \equiv \dot{K}_i(t)$.[5] From what has previously been said, the production possibilities set at time t can be expressed in the form $S(K(t))$. The consumption-investment pair (C,I) is producible at time t if and only if

$$(C,I) \ \epsilon \ S(K(t)). \tag{1}$$

Let p_i represent the price of investment good i relative to a consumption price of unity. A real net national product function (with consumption as numeraire) could be defined as follows:

$$Y(K,p) \equiv \max_{(C,I)\epsilon S(K)} [C + pI]. \tag{2}$$

While we could get by on much weaker assumptions, it will be postulated here that $\frac{\partial Y}{\partial K_i}$ and $\frac{\partial Y}{\partial p_i}$ exist for all K and $p \geq 0$.

A <u>feasible</u> trajectory $\{C(t),K(t)\}$ is one satisfying for all $t \geq 0$

$$(C(t),\dot{K}(t)) \ \epsilon \ S(K(t)), \tag{3}$$

$$K(0) = \bar{K}_o, \tag{4}$$

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[5] As usual, $\dot{K}_i \equiv \frac{dK_i}{dt}$, and $K \equiv (K_i)$.

where $\bar{K}_o$ is the original endowment of capital which is available at starting time $t = 0$.

Generally speaking there are an infinite number of feasible trajectories. We can narrow them down to a single family by presuming a competitive like economy with a fixed rate of return on consumption equal to r. A <u>competitive</u> trajectory $\{C^*(t),K^*(t)\}$ is any feasible trajectory for which there exists a set of investment prices $\{p(t)\}$ such that for all $t \geq 0$

$$Y(K^*(t),p(t)) = C^*(t) + p(t)\dot{K}^*(t) \tag{5}$$

$$\left.\frac{\partial Y}{\partial K_i}\right|_* = rp_i(t) - \dot{p}_i(t) \qquad i = 1,\ldots,n. \tag{6}$$

Equation (5) just states that what is actually produced by the economy at any time maximizes its income - in other words relative prices are equal to marginal rates of transformation. Condition (6) is the well-known intertemporal efficiency condition of a competitive capital market with perfect foresight.[6]

Actually, equations (5) and (6) are necessary Pontryagin-type conditions[7]

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[6]   If I buy an extra unit of capital good i it will cost me p_i, whereas I can use it to produce $\frac{\partial Y}{\partial K_i}$ worth of output and sell it for $p_i + \dot{p}_i$ one period later. In a competitive capital market, we must therefore have $p_i = \frac{\partial Y}{\partial K_i} + \frac{(p_i + \dot{p}_i)}{1 + r}$, which reduces to equation (6). For a more rigorous explanation, see [2], p. 318. The * symbol in equation (6) means that the derivative is evaluated (at time t) along a competitive trajectory.

[7]   See, for example, [4] or [1].

for any solution of the optimal control problem:[8]

$$\text{maximize} \qquad \int_0^\infty C(t) \, e^{-rt} \, dt \qquad (7)$$

$$\text{subject to} \qquad (C(t), \dot{K}(t)) \, \epsilon \, S(K(t)) \qquad (8)$$

$$K(0) = \bar{K}_0 \quad . \qquad (9)$$

What we have been calling net national product is just the Hamiltonian for a general optimization problem of the form (7) - (9). Therefore, an alternative but equivalent approach to the one taken in this paper would be to ask what the Hamiltonian along an optimal growth path measures.

3. What is Net National Product?

Even granted that consumption is the ultimate end of economic activity, the national income statistician's practice of adding in investment goods to the value of consumption by weighting them with prices measuring their marginal rates of transformation might still be defended as a measure of the economy's power to consume at a constant rate. After all, a standard

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[8] Sometimes it is argued that the maximand of an optimal growth problem should have the form (7) (once it has been determined that $C(t)$ is the appropriate measure of satisfaction at time t) because it is reasonable to require that the future liik the same from any initial time point; see for example [6]. Note that it is trivial to formally force any technology into the nominally time independent form (8) merely by introducing a new artificial "capital good" K_{n+1} which is a proxy for time and satisfies the differential equation $\dot{K}_{n+1} = 1$; the difficulty with such a trick from a strictly economic point of view is that it is not clear where the investment $p_{n+1} \dot{K}_{n+1} = p_{n+1}$ (which measures the value of time per se) shows up in the national income accounting of any real economy.

welfare interpretation of NNP is that it is the largest permanently main-
tainable value of consumption. If all investment were convertable into
consumption at the given price-transformation rates, the maximum attainable
level of consumption which could be maintained forever without running down
capital stocks would appear to be NNP as conventionally measured by $C^* + p\dot{K}^*$.

Unfortunately, such reasoning is insufficient because <u>marginal</u> trans-
formation rates cannot in general be used to change <u>non-marginal</u> amounts of
investment into consumption. For this reason the <u>consumption</u> level $C^*(t) +$
$p(t)\dot{K}^*(t)$ is undoubtedly not attainable at time t. Such a situation is de-
picted in Fig. 1 for case n = 1. The economy is located at point A on the
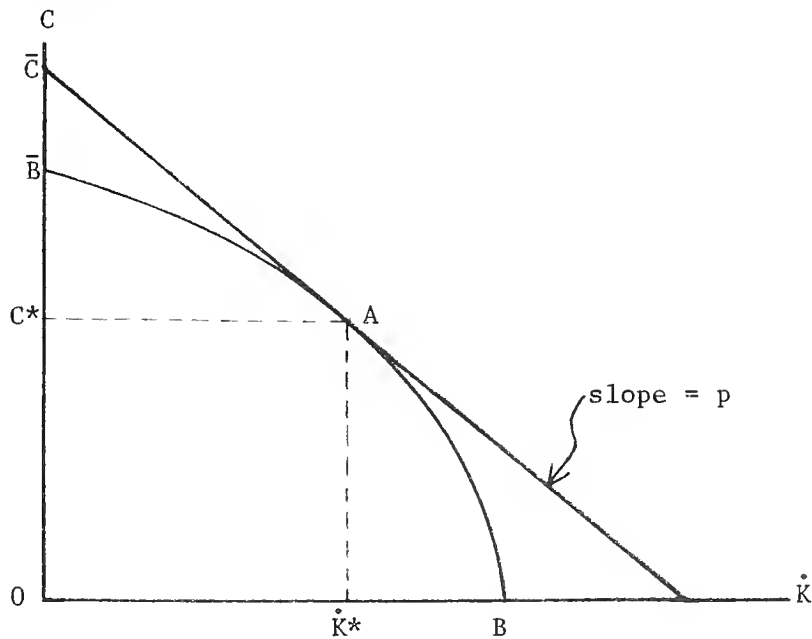production possibilities frontier $\bar{B}B$. Real net national product $C^* + p\dot{K}^*$

Figure 1

is geometrically represented as $O\bar{C}$. But $O\bar{C}$ is a strictly <u>hypothetical</u>
consumption level at the present time since the largest permanently main-
tainable level of consumption which can <u>actually</u> be obtained is $O\bar{B}$. The
two are equal only if the production possibilities frontier is a linear
surface.

All this notwithstanding, it turns out that the maximum welfare actually attainable from time t on along a competitive trajectory,

$$\int_t^\infty C*(s) \ e^{-r(s-t)} \ ds,$$

is exactly the same as what <u>would</u> be obtained from the <u>hypothetical</u> constant consumption level $C*(t) + p(t)\dot{K}*(t)$. In this sense the naive interpretation of the current power to consume at a constant rate idea gives the right answer, although for the wrong reason. Net national product is what might be called the <u>stationary equivalent</u> of future consumption, and this is its primary welfare interpretation.[9]

4. <u>Proof of the Main Proposition</u>

We want to show that along a competitive trajectory

$$\int_t^\infty [C*(t) + p(t)\dot{K}*(t)] \ e^{-r(s-t)} ds = \int_t^\infty C*(s) \ e^{-r(s-t)} ds,$$

or that

$$Y*(t) = r\int_t^\infty C*(s) \ e^{-r(s-t)} ds, \tag{10}$$

where

$$Y*(t) \equiv Y(K*(t), p(t)) = C*(t) + p(t)\dot{K}*(t). \tag{11}$$

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[9] If there is non-attributable or "atmospheric" technical change, national product will be less than the stationary equivalent of consumption by a term measuring the present discounted value of the pure effect of time alone on increasing output.

This is easily shown as follows. First differentiate Y* totally with respect to time:

$$\dot{Y}*(t) = \sum_1^n \left.\frac{\partial Y}{\partial K_i}\right|_* \dot{K}_i^*(t) + \sum_1^n \left.\frac{\partial Y}{\partial P_i}\right|_* \dot{p}_i(t). \tag{12}$$

From (2), (5), and the theory of cost functions,[10]

$$\left.\frac{\partial Y}{\partial P_i}\right|_* = \dot{K}_i^*(t). \tag{13}$$

Substituting from (6) and (13) into (12) and cancelling out the term $\sum_1^n \dot{p}_i(t) \dot{K}_i^*(t)$ yields

$$\dot{Y}*(t) = r \sum_1^n p_i(t) \dot{K}_i^*(t),$$

which by (11) is equivalent to

$$\dot{Y}*(t) = r(Y*(t) - C*(t)). \tag{14}$$

It can easily be verified that the solution to the differential equation (14) is given by (10), the proposition to be proved.

5. Unanticipated Technological Change; an Application

So far we have in effect been using the law of large numbers to abstract away the sporadic or stochastic nature of technical change. Instead, it has been attributed in regular fashion (ultimately) to capital accumulation of one form or another (including educational investment in human

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[10] See, for example [3], ch. 1.

capital and research investment in increased stocks of knowledge). Now we will go to the opposite extreme in asking what happens if an unanticipated invention or innovation is suddenly and unexpectedly discovered.[11] Such "windfall" technical change can be expressed analytically as an enlargement of the production possibilities set from $S(K)$ to $\hat{S}(K)$ where

$$\hat{S}(K) \supseteq S(K). \qquad (15)$$

For notational convenience and without loss of generality, this unexpected change is presumed to occur at time zero. In a perfect foresight competitive economy, a new competitive trajectory $\{\hat{C}*(t), \hat{K}*(t)\}$ will come into being with a new set of prices $\hat{p}(t)$ obeying conditions just like (2) - (6), the only substantive difference being that $\hat{S}(K)$ now replaces $S(K)$. From (15) and the sufficiency[12] of conditions (2) - (6) for describing a solution to (7) - (9), it follows that

$$\int_0^\infty \hat{C}*(t) \; e^{-rt} \; dt \; > \; \int_0^\infty C*(t) \; e^{-rt} \; dt.$$

This means, along with (10), (11), that

$$\hat{C}*(0) + \hat{p}(0)\dot{\hat{K}}*(0) = r\int_0^\infty \hat{C}*(t) \; e^{-rt} \; dt \; > \; r\int_0^\infty C*(t) \; e^{-rt} dt = C*(0) + p(0)\dot{K}*(0).$$

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[11] Alternatively, we could consider the situation where a seemingly forseen invention or innovation failed to materialize after performing the necessary research which was expected to uncover it.

[12] Actually, for sufficiency to hold we would need something additional like convexity of S and fulfillment of a transversality condition. See, e.g., [1] or [7].

In other words, the welfare effect on present discounted consumption of an unexpected windfall gain in production possibilities should exactly and immediately be capitalized by a sudden rise in net national product.[13] This is true even if the technological discovery is of the embodied kind which cannot increase current productive capacity but will be helpful in adding to production possibilities only at some time in the future, after the necessary capital has been accumulated.

To see this point more clearly, consider a simple two good durable capital model of the form

$$g(C,\dot{K}) = f(K).$$

The function $g(C,\dot{K})$ (which is concave and homogeneous of degree one) gives the curved production possibilities between C and $\dot{K}$ for a given value of K, whereas $f(K)$ (which exhibits diminishing returns) is in some sense a measure of the overall capacity of the economy as a function of its capital stock. In Figure 2, $f(K)$ is depicted before and after a windfall technological
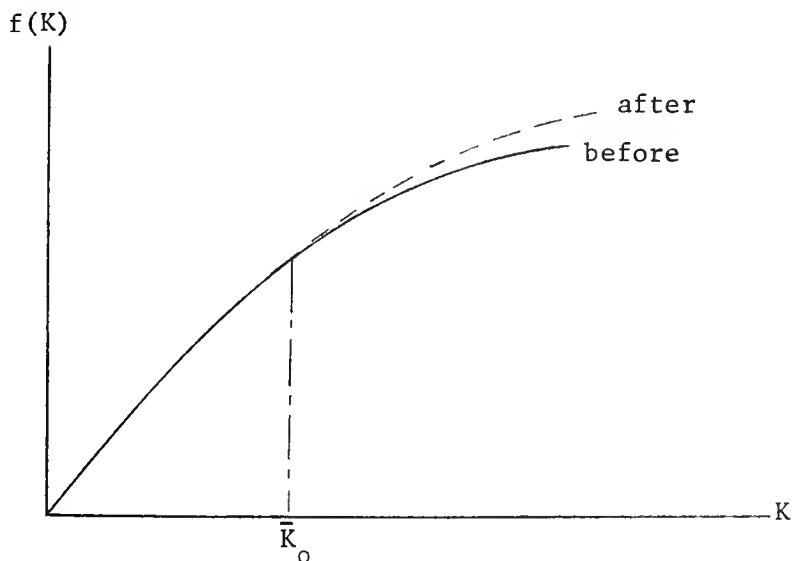

Figure 2

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[13] Naturally a windfall decrease in production possibilities (like the discovery that petroleum reserves are less than expected) leads to a decline in current NNP which reflects the extent of the loss.

advance which makes capital intensive techniques more productive than was
previously thought possible. One interpretation of the fact that f(K) does
not shift upward for values of $K \leq \bar{K}_0$ (the present endowment of capital) is
that the discovery has to be embodied in new capital. Such an innovation
has no effect on the current production possibilities surface

$$\{C,\dot{K}/ \ g(C,\dot{K}) \ = \ f(\bar{K}_0)\},$$

depicted as $\bar{B}B$ in Figure 3. Suppose that before the unanticipated change
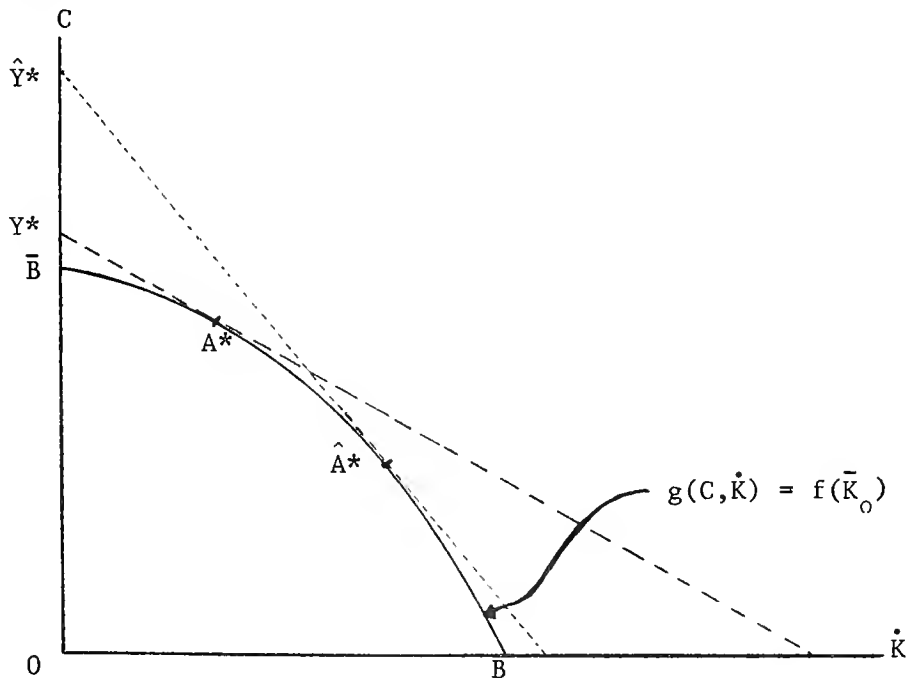
Figure 3

the economy was operating at point A* with NNP = OY*. We know that after
the unexpected favorable shift in f(K), current NNP must increase, say to
$O\hat{Y}*$. It follows that the economy must move to a point like $\hat{A}*$. Thus, at
the time of its discovery unanticipated capital embodied technical change
should, other things being equal, result in less consumption and more in-
vestment. An index number version of this result exists for the case of
more than one capital good.

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